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Typical upper L^q-dimensions of measures for $q \in [0, 1]$

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Abstract

For a probability measure μ on a subset of $\mathbb{R}^d$, the lower and upper L^q-dimensions of order $q \in \mathbb{R}$ are defined by

$$\underline{D}_\mu(q) = \liminf_{r \searrow 0} \frac{\log \int \mu(B(x,r))^{q-1}\, d\mu(x)}{-\log r},$$
$$\overline{D}_\mu(q) = \limsup_{r \searrow 0} \frac{\log \int \mu(B(x,r))^{q-1}\, d\mu(x)}{-\log r}.$$

In previous work we studied the typical behaviour (in the sense of Baire's category) of the L^q-dimensions $\underline{D}_\mu(q)$ and $\overline{D}_\mu(q)$ for $q \geqslant 1$. In the present work we study the typical behaviour (in the sense of Baire's category) of the upper L^q-dimensions $\overline{D}_\mu(q)$ for $q \in [0, 1]$.

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1. Statement of results

Let K be a compact subset of $\mathbb{R}^d$. For a probability measure μ on K, the L^q-dimensions of μ are defined as follows. For $r > 0$ and a real number q, write

$$I_\mu(r;q) = \int \mu\big(B(x,r)\big)^{q-1}\, d\mu(x). \tag{1.1}$$

The lower and upper L^q-dimensions of order q are now defined by

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$$\underline{D}_\mu(q) = \liminf_{r \searrow 0} \frac{\log I_\mu(r;q)}{-\log r},$$
$$\overline{D}_\mu(q) = \limsup_{r \searrow 0} \frac{\log I_\mu(r;q)}{-\log r}. \tag{1.2}$$

The main significance of the L^q-dimensions is their relationship with the multifractal spectrum of μ. For a probability measure μ on $\mathbb{R}^d$ (or on a general metric space), the local dimension of μ at the point x is defined by

$$\dim_{\mathrm{loc}}(x;\mu) = \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r}.$$

We define the Hausdorff multifractal spectrum function, f_μ, of μ as the Hausdorff dimension of the level sets of the local dimension of μ, i.e. we put

$$f_\mu(\alpha) = \dim\left\{ x \in \mathbb{R}^d \,\middle|\, \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\}, \quad \alpha \geqslant 0, \tag{1.3}$$

where dim denotes the Hausdorff dimension. Next, recall that the Legendre transform φ^* of a function $\varphi : \mathbb{R} \to \mathbb{R}$ is defined by $\varphi^*(x) = \inf_y (xy + \varphi(y))$. In the 1980s it was conjectured in the physics literature [5,6] that for "good" measures the following result, relating the multifractal spectrum function f_μ to the Legendre transform of the L^q-dimensions, holds: namely (1) that the L^q-dimensions coincide, and (2) that the multifractal spectrum function f_μ coincide with the value of the Legendre transform of the L^q-dimensions, i.e.

$$\underline{D}_\mu(q) = \overline{D}_\mu(q), \tag{1.4}$$

and

$$\dim\left\{ x \in K \,\middle|\, \lim_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \right\} = \underline{D}_\mu^*(\alpha) = \overline{D}_\mu^*(\alpha), \tag{1.5}$$

for all $q \in \mathbb{R}$ and all $\alpha \geqslant 0$. This result is known as the Multifractal Formalism. During the 1990s there has been an enormous interest in verifying the Multifractal Formalism and computing the multifractal spectra of measures in the mathematical literature, and within the last 8 or 9 years the multifractal spectra of various classes of measures in Euclidean space $\mathbb{R}^d$ exhibiting some degree of self-similarity have been computed rigorously, cf. [2] and the references therein.

In this paper we study the L^q-dimensions of a typical measure in the sense of Baire. For a compact subset K of $\mathbb{R}^d$, we denote the family of Borel probability measures on K by $\mathcal{P}(K)$ and we equip $\mathcal{P}(K)$ with the weak topology. We will now say that a typical probability measure on K has property P, if the set of probability measures that do not have property P, i.e. if the set

$$\{\mu \in \mathcal{P}(K) \mid \mu \text{ does not have property P}\},$$

is of the first category with respect to the weak topology on $\mathcal{P}(K)$. The typical behaviour of various other quantities related to multifractal analysis has also been studied. In particular, the local dimension $\dim_{\mathrm{loc}}(x;\mu)$ of a typical measure has been studied by Haase [4] and investigated further by Genyuk [3]. We also note that the multifractal spectrum of a typical continuous function has been studied by several authors, cf. [1,7,8].

In [9] we found the L^q-dimensions of a typical measure for $q \geqslant 1$, and the purpose of this paper is to complement this result by determining the L^q-dimensions of a typical measure for $q \in [0,1]$. However, before we state this result it is instructive to recall the result from [9] giving the L^q-dimensions of a typical measure for $q \geqslant 1$. To state this result we begin with a few definitions. For a subset E of $\mathbb{R}^d$, we denote the lower box dimension of E and the upper box dimension of E by $\underline{\dim}_{\mathrm{B}}(E)$ and $\overline{\dim}_{\mathrm{B}}(E)$, respectively; the reader is referred to [2] for the definitions of the box dimensions. Also, for a subset K of $\mathbb{R}^d$ and $x \in K$ we define the lower local box dimension of K at x and the upper local box dimension of K at x by

$$\underline{\dim}_{\mathrm{B,loc}}(x, K) = \lim_{r \searrow 0} \underline{\dim}_{\mathrm{B}}\big(K \cap B(x, r)\big)$$

and

$$\overline{\dim}_{\mathrm{B,loc}}(x, K) = \lim_{r \searrow 0} \overline{\dim}_{\mathrm{B}}\big(K \cap B(x, r)\big),$$

respectively. We can now state the result from [9] giving the L^q-dimensions of a typical measure for $q \geqslant 1$.

Theorem A. *(See [9].) Let K be a compact subset of $\mathbb{R}^d$. Write*

$$\underline{s} = \inf_{x \in K} \underline{\dim}_{\mathrm{B,loc}}(x, K),$$
$$\bar{s} = \inf_{x \in K} \overline{\dim}_{\mathrm{B,loc}}(x, K),$$
$$s = \overline{\dim}_{\mathrm{B}}(K).$$

Observe that $\underline{s} \leqslant \bar{s} \leqslant s$. Assume that $\underline{s} = \bar{s} = s$. (*This condition is clearly satisfied if, for example, K is the closure of an open and bounded set or if K is a self-similar set satisfying the open set condition.*)

(1) *For all measures $\mu \in \mathcal{P}(K)$ we have*

$$s(1-q) \leqslant \underline{D}_\mu(q) \leqslant \overline{D}_\mu(q) \leqslant 0$$

for all $q \geqslant 1$.

(2) *A typical measure $\mu \in \mathcal{P}(K)$ satisfies the following*

$$\underline{D}_\mu(q) = s(1-q),$$
$$\overline{D}_\mu(q) = 0,$$

for all $q \geqslant 1$.

As mentioned earlier, the purpose of this paper is to complement Theorem A by determining the upper L^q-dimensions of a typical measure for $q \in [0, 1]$. Indeed, we obtain the following result.

Theorem 1. *Let K be a compact subset of $\mathbb{R}^d$. Let $\underline{s}$, $\bar{s}$ and s be defined as in Theorem A.*

(1) *For all measures $\mu \in \mathcal{P}(K)$ we have*

$$0 \leqslant \overline{D}_\mu(q) \leqslant s(1-q)$$

for all $q \in [0, 1]$.

(2) *A typical measure $\mu \in \mathcal{P}(K)$ satisfies the following*

$$\underline{s}(1-q) \leqslant \overline{D}_\mu(q) \leqslant \bar{s}(1-q),$$

for all $q \in [0, 1]$. *In particular, if* $\underline{s} = \bar{s} = s$ (*this condition is clearly satisfied if, for example,* K *is the closure of an open and bounded set or if* K *is a self-similar set satisfying the open set condition*), *then a typical measure* $\mu \in \mathcal{P}(K)$ *satisfies the following*

$$\overline{D}_\mu(q) = s(1-q)$$

for all $q \in [0, 1]$.

The proof of Theorem 1 is given in Section 2. It is clearly unfortunate that we have not been able to determine the lower L^q-dimension of a typical measure for $q \in [0, 1]$. However, we believe that the lower L^q-dimension of a typical measure equals 0 for all $q \in [0, 1]$ and make the following conjecture

Conjecture 2. *Let* K *be a compact subset of* $\mathbb{R}^d$. *A typical measure* $\mu \in \mathcal{P}(K)$ *satisfies the following*

$$\underline{D}_\mu(q) = 0$$

for all $q \in [0, 1]$.

We also make the following conjecture regarding the L^q-dimensions of a typical measure for $q < 0$.

Conjecture 3. *Let* K *be a compact subset of* $\mathbb{R}^d$. *A typical measure* $\mu \in \mathcal{P}(K)$ *satisfies the following*

$$\underline{D}_\mu(q) = 0,$$
$$\overline{D}_\mu(q) = \infty$$

for all $q < 0$.

2. Proof of Theorem 2

2.1. The sets Δ^{l} *and* Δ^{u}

Write

$$\begin{aligned} \Delta^{\mathrm{l}} &= \bigl\{\mu \in \mathcal{P}(K) \mid \underline{s}(1-q) \leqslant \overline{D}_\mu(q) \text{ for all } q \in [0,1]\bigr\}, \\ \Delta^{\mathrm{u}} &= \bigl\{\mu \in \mathcal{P}(K) \mid \overline{D}_\mu(q) \leqslant \bar{s}(1-q) \text{ for all } q \in [0,1]\bigr\}. \end{aligned} \tag{2.1}$$

We must prove that the two sets Δ^{l} and Δ^{u} are residual. In Section 2.2 we prove that the set Δ^{l} is residual, and finally in Section 2.3 we prove that the set Δ^{u} is residual.

It is well-known (cf., for example, [10, p. 51, Theorem 6.8]) that the weak topology on $\mathcal{P}(K)$ is induced by the metric L on $\mathcal{P}(K)$ defined as follows. Let $\mathrm{Lip}(K)$ denote the family of Lipschitz functions $f : K \to \mathbb{R}$ with $|f| \leqslant 1$ and $\mathrm{Lip}(f) \leqslant 1$ where $\mathrm{Lip}(f)$ denotes the Lipschitz constant of f. The metric L is now defined by

$$L(\mu, \nu) = \sup_{f \in \mathrm{Lip}(K)} \left| \int f \, d\mu - \int f \, d\nu \right|$$

for $\mu, \nu \in \mathcal{P}(K)$. We will always equip $\mathcal{P}(K)$ with the metric L and all balls in $\mathcal{P}(K)$ will be with respect to the metric L, i.e. if $\mu \in \mathcal{P}(K)$ and $r > 0$ we will write $B(\mu, r) = \{\nu \in \mathcal{P}(K) \mid L(\mu, \nu) < r\}$ for the ball with centre at μ and radius equal to r. For $x \in K$ and $r > 0$, define $f_{x,r} : K \to \mathbb{R}$ by

$$f_{x,r}(t) = \begin{cases} r & \text{if } |x - t| \leqslant r; \\ -|t - x| + 2r & \text{if } r < |x - t| < 2r; \\ 0 & \text{if } 2r \leqslant |x - t|. \end{cases} \tag{2.2}$$

Observe that if $r \leqslant 1$, then $f_{x,r}$ is Lipschitz with $|f_{x,r}| \leqslant 1$ and $\mathrm{Lip}(f_{x,r}) = 1$. In particular, this implies that if $r \leqslant 1$, then

$$\left| \int f_{x,r} \, d\mu - \int f_{x,r} \, d\nu \right| \leqslant L(\mu, \nu) \tag{2.3}$$

for all $\mu, \nu \in \mathcal{P}(K)$. This inequality will be used frequently in Sections 2.2 and 2.3.

2.2. *The set Δ^1 is residual*

In this section we prove that the set Δ^1 is residual. For a real number t write

$$\Delta^1_t = \left\{ \mu \in \mathcal{P}(K) \mid t(1 - q) \leqslant \overline{D}_\mu(q) \text{ for all } q \in [0, 1] \right\}.$$

Since

$$\Delta^1 = \bigcap_{\substack{t \in \mathbb{Q} \\ t < \underline{s}}} \Delta^1_t,$$

it clearly suffices to prove that the set Δ^1_t is residual for each rational number t with $t < \underline{s}$. Therefore fix a rational number t with $t < \underline{s}$. To prove that the set Δ^1_t is residual it clearly suffices to construct a set $M^1 \subseteq \mathcal{P}(K)$ satisfying the following three conditions:

(1) $M^1 \subseteq \Delta^1_t$;
(2) M^1 is dense in $\mathcal{P}(K)$;
(3) M^1 is $\mathcal{G}_\delta$.

Lemma 2.2.1. *Assume that $x_0 \in K$, $r_0 > 0$ and $t \geqslant 0$ satisfy*

$$t < \underline{\dim}_{\mathrm{B}}\big(K \cap B(x_0, r_0)\big).$$

Then there exists $c > 0$ such that for each $r > 0$ there exists a measure $\mu \in \mathcal{P}(K)$ with

(1) $\operatorname{supp} \mu \subseteq K \cap B(x_0, r_0)$;
(2) *for all $x \in K$ we have $\mu(B(x, r)) \leqslant cr^t$.*

Proof. For $r > 0$, let $M_r(K \cap B(x_0, r_0))$ denote the largest number of pairwise disjoint balls of radius r with centres in $K \cap B(x_0, r_0)$. Then

$$\underline{\dim}_{\mathrm{B}}\big(K \cap B(x_0, r_0)\big) = \liminf_{r \searrow 0} \frac{\log M_r(K \cap B(x_0, r_0))}{-\log r},$$

cf. [2]. We can thus find $0 < \delta \leqslant 1$ such that $\frac{\log M_r(K \cap B(x_0, r_0))}{-\log r} > t$ for all $0 < r \leqslant \delta$, whence

$$M_r\big(K \cap B(x_0, r_0)\big) > r^{-t} \tag{2.4}$$

for all $0 < r \leqslant \delta$. Now put $c = \frac{1}{\delta^t} \geqslant 1$. We must prove that for each $r > 0$ there exists a measure $\mu \in \mathcal{P}(K)$ satisfying conditions (1) and (2). Therefore fix $r > 0$. We divide the proof into two cases.

Case 1: $\delta < r$. Pick any $\mu \in \mathcal{P}(K)$ with $\operatorname{supp}\mu \subseteq K \cap B(x_0, r_0)$. (For example, we may put $\mu = \delta_{x_0}$.) For all $x \in K$ we clearly have $\mu(B(x,r)) \leqslant 1 = c\delta^t < cr^t$.

Case 2: $0 < r \leqslant \delta$. For brevity write $M = M_r(K \cap B(x_0, r_0))$. By definition of M there exist M pairwise disjoint balls $B(x_1, r), \ldots, B(x_M, r)$ with centres $x_1, \ldots, x_M$ in $K \cap B(x_0, r_0)$. Now put $\mu = \frac{1}{M}\sum_{i=1}^{M} \delta_{x_i}$. Then clearly $\operatorname{supp}\mu \subseteq K \cap B(x_0, r_0)$. Next, let $x \in K$ and observe that the ball $B(x,r)$ can at most contain one of the x_i's. Indeed, otherwise there exist two distinct indices i and j such that $x_i, x_j \in B(x,r)$, whence $x \in B(x_i, r) \cap B(x_j, r)$, contradicting the fact that the balls $B(x_1, r), \ldots, B(x_M, r)$ are pairwise disjoint. Since $r \leqslant \delta$ and the ball $B(x,r)$ contain at most one of the x_i's, we conclude from (2.4) that

$$\mu\big(B(x,r)\big) \leqslant \frac{1}{M} = \frac{1}{M_r(K \cap B(x_0, r_0))} < r^t \leqslant cr^t.$$

This completes the proof of Lemma 2.2.1. □

Let $(x_n)_n$ be a dense sequence in K. Fix n and $i = 1, \ldots, n$. Since

$$t < \underline{s} = \inf_{x \in K} \underline{\dim}_{\mathrm{B,loc}}(x, K) \leqslant \underline{\dim}_{\mathrm{B}}\left(K \cap B\left(x_i, \frac{1}{n}\right)\right),$$

it follows from Lemma 2.2.1 that there exists a constant $c_{n,i}$ such that for all $r > 0$ there exists a measure $\mu \in \mathcal{P}(K)$ with

(1) $\operatorname{supp}\mu \subseteq K \cap B(x_i, \frac{1}{n})$;
(2) for all $x \in K$ we have $\mu(B(x,r)) \leqslant c_{n,i} r^t$.

Now put $c_n = \max(2^t c_{n,1}, \ldots, 2^t c_{n,n}, n)$ and $r_n = \frac{1}{e^{c_n}}$. We can thus choose a measure $\mu_{n,i} \in \mathcal{P}(K)$ with

(1) $\operatorname{supp}\mu_{n,i} \subseteq K \cap B(x_i, \frac{1}{n})$;
(2) for all $x \in K$ we have $\mu_{n,i}(B(x, 2r_n)) \leqslant c_{n,i}(2r_n)^t$.

For a positive integer n write

$$\Lambda_n^1 = \left\{ \sum_{i=1}^{n} p_i \mu_{n,i} \,\middle|\, p_i \geqslant 0, \sum_{i=1}^{n} p_i = 1 \right\}.$$

Next put

$$G_n^1 = \bigcup_{\lambda \in \Lambda_n^1} B(\lambda, r_n^{t+1}),$$

and define the set $M^1 \subseteq \mathcal{P}(K)$ by

$$M^1 = \bigcap_m \bigcup_{n \geqslant m} G_n^1.$$

Below we show that the set M^1 has the following three properties: (1) $M^1 \subseteq \Delta_t^1$, (2) M^1 is dense in $\mathcal{P}(K)$, and (3) M^1 is $\mathcal{G}_\delta$. The set M^1 is clearly $\mathcal{G}_\delta$, and it thus suffices to show that $M^1 \subseteq \Delta_t^1$ and that M^1 is dense in $\mathcal{P}(K)$. This is done in Proposition 2.2.2 and Lemma 2.2.4.

Proposition 2.2.2. *We have $M^1 \subseteq \Delta_t^1$.*

Proof. Let $\mu \in M^1$ and fix a positive integer m. Since $\mu \in M^1$, there exists $n \geqslant m$ and a measure $\lambda \in \Lambda_n^1$ such that $L(\mu, \lambda) \leqslant r_n^{t+1}$. Also, since $\lambda \in \Lambda_n^1$, we can find $p_1, \ldots, p_n$ with $p_i \geqslant 0$ and $\lambda = \sum_i p_i \mu_{n,i}$. Now observe that for all $x \in K$ we have (using (2.3))

$$\begin{aligned}\mu\big(B(x, r_n)\big) &= \int 1_{B(x,r_n)}\, d\mu \leqslant \int \frac{f_{x,r_n}}{r_n}\, d\mu \leqslant \frac{1}{r_n}\left(L(\mu, \lambda) + \int f_{x,r_n}\, d\lambda\right)\\ &\leqslant \frac{1}{r_n}\Big(L(\mu, \lambda) + r_n \lambda\big(B(x, 2r_n)\big)\Big) \leqslant \frac{1}{r_n}\left(r_n^{t+1} + r_n \sum_i p_i \mu_{n,i}\big(B(x, 2r_n)\big)\right)\\ &\leqslant \frac{1}{r_n}\left(r_n^{t+1} + r_n \sum_i p_i c_{n,i} (2r_n)^t\right) \leqslant \frac{1}{r_n}\left(r_n^{t+1} + r_n \sum_i p_i c_n r_n^t\right)\\ &= (1 + c_n) r_n^t.\end{aligned}$$

This implies that

$$I_\mu(r_n; q) = \int \mu\big(B(x, r_n)\big)^{q-1}\, d\mu(x) \geqslant (1 + c_n)^{q-1} r_n^{t(q-1)}$$

for all $q \in [0, 1]$, whence

$$\begin{aligned}\overline{D}_\mu(q) &= \limsup_{r \searrow 0} \frac{\log I_\mu(r; q)}{-\log r} \geqslant \limsup_n \frac{\log I_\mu(r_n; q)}{-\log r_n}\\ &\geqslant \limsup_n (q-1) \frac{\log(1 + c_n) + t \log r_n}{-\log r_n}\\ &= \limsup_n (q-1) \left(\frac{\log(1 + c_n)}{c_n} - t\right) = t(1-q)\end{aligned}$$

for all $q \in [0, 1]$. □

Lemma 2.2.3. *Let $F \subseteq \mathbb{R}^d$ be a bounded Borel set and $r > 0$. Then there exists finitely many pairwise disjoint Borel sets $F_1, \ldots, F_N$ with* diam $F_j \leqslant r$ *such that $F \subseteq \bigcup_j F_j$, and such that for each j, there exists an $x_j \in F$ satisfying*

$$B\left(x_j, \frac{r}{4}\right) \subseteq F_j.$$

Proof. First construct a sequence of balls $B(x_1, \frac{r}{2}), B(x_2, \frac{r}{2}), \ldots$ such that $x \in F$ and $|x_i - x_j| > \frac{r}{2}$ for all $i \neq j$. Because F is totally bounded this process must terminate at some finite stage, giving balls $B(x_1, \frac{r}{2}), B(x_2, \frac{r}{2}), \ldots, B(x_N, \frac{r}{2})$ such that any $x \in F$ must satisfy $\min_j |x - x_j| \leqslant \frac{r}{2}$ (and consequently $F \subseteq \bigcup_{j=1}^N B(x_j, \frac{r}{2})$). Note that the smaller balls $B(x_1, \frac{r}{4}), B(x_2, \frac{r}{4}), \ldots, B(x_N, \frac{r}{4})$ are pairwise disjoint. Now set

$$F_1 = B\left(x_1, \frac{r}{2}\right) \Big\backslash \bigcup_{i=2}^N B\left(x_i, \frac{r}{4}\right),$$

$$F_j = B\left(x_j, \frac{r}{2}\right) \Big\backslash \left(\bigcup_{i=1}^{j-1} F_i \cup \bigcup_{i=j+1}^N B\left(x_i, \frac{r}{4}\right)\right) \quad \text{for } j = 2, \ldots, N-1,$$

$$F_N = B\left(x_N, \frac{r}{2}\right) \setminus \bigcup_{i=1}^{N-1} F_i.$$

It is clear that the sets $F_1, F_2, \ldots, F_N$ are pairwise disjoint, and since $B(x_1, \frac{r}{4}), B(x_2, \frac{r}{4}), \ldots, B(x_N, \frac{r}{4})$ are pairwise disjoint we conclude that $B(x_j, \frac{r}{4}) \subseteq F_j$ and $F \subseteq \bigcup_j F_j$. $\square$

Lemma 2.2.4. *The set M^1 is dense in $\mathcal{P}(K)$.*

Proof. Since $\mathcal{P}(K)$ is a complete metric space (because K is compact) and each set $\bigcup_{k \geqslant m} G_k^1$ is open, it suffices (by Baire's Theorem) to show that $\bigcup_{k \geqslant m} G_k^1$ is dense for all m. In order to show that $\bigcup_{k \geqslant m} G_k^1$ is dense it suffices to show that the subset $\bigcup_{k \geqslant m} \Lambda_k^1$ is dense for all m. Therefore fix a positive integer m. Let $\mu \in \mathcal{P}(K)$ and $0 < \varepsilon \leqslant 1$. According to Lemma 2.2.3 we may choose finitely many pairwise disjoint Borel sets $K_1, \ldots, K_N$ with $\operatorname{diam} K_j \leqslant \varepsilon$ such that $K \subseteq \bigcup_j K_j$, and such that for each j there exists an $y_j \in K$ satisfying

$$B\left(y_j, \frac{\varepsilon}{4}\right) \subseteq K_j.$$

Since the sequence $(x_k)_k$ is dense in K we can also choose a positive integer $n \geqslant m$ such that $\frac{1}{n} \leqslant \frac{\varepsilon}{8}$ and $\{x_1, \ldots, x_n\} \cap B(y_j, \frac{\varepsilon}{8}) \neq \emptyset$ for all j. Hence, for each $j = 1, \ldots, N$ we can pick a (not necessarily unique) $i(j)$ with

$$x_{i(j)} \in B\left(y_j, \frac{\varepsilon}{8}\right).$$

Now put

$$p_i = \begin{cases} \mu(K \cap K_j) & \text{if } i = i(j) \text{ for some } j = 1, \ldots, N; \\ 0 & \text{if } i \neq i(j) \text{ for all } j = 1, \ldots, N. \end{cases}$$

Finally, write

$$\lambda = \sum_i p_i \mu_{n,i}.$$

We will now show that $\lambda \in \bigcup_{k \geqslant m} \Lambda_k^1$ and that $L(\mu, \lambda) \leqslant \varepsilon$. Indeed, we clearly have that $\lambda \in \Lambda_n^1 \subseteq \bigcup_{k \geqslant m} \Lambda_k^1$. Next, we prove that $L(\mu, \lambda) \leqslant \varepsilon$. We have

$$L(\mu, \lambda) = \sup_{f \in \operatorname{Lip}(K)} \left| \int f\, d\mu - \int f\, d\lambda \right| \leqslant \sup_{f \in \operatorname{Lip}(K)} \sum_j \left| \int_{K \cap K_j} f\, d\mu - \int_{K \cap K_j} f\, d\lambda \right|. \tag{2.5}$$

First, observe that if $f : K \to \mathbb{R}$ is a real valued function with $\operatorname{Lip}(f) \leqslant 1$ and $|f| \leqslant 1$, then

$$\mu(K \cap K_j) \inf_{x \in K \cap K_j} f(x) \leqslant \int_{K \cap K_j} f\, d\mu \leqslant \mu(K \cap K_j) \sup_{x \in K \cap K_j} f(x). \tag{2.6}$$

Next, observe that since $\operatorname{supp} \mu_{n,i(j)} \subseteq K \cap B(x_{i(j)}, \frac{1}{n}) \subseteq K \cap B(x_{i(j)}, \frac{\varepsilon}{8}) \subseteq K \cap B(y_j, \frac{\varepsilon}{4}) \subseteq K \cap K_j$ and the sets $K_1, \ldots, K_N$ are pairwise disjoint, we have

$$\int_{K \cap K_j} f\, d\lambda = p_{i(j)} \int_{K \cap K_j} f\, d\mu_{n,i(j)}.$$

It follows from this that

$$\int_{K\cap K_j} f\,d\lambda \leqslant p_{i(j)}\mu_{n,i(j)}(K\cap K_j)\sup_{x\in K\cap K_j} f(x) = \mu(K\cap K_j)\sup_{x\in K\cap K_j} f(x), \tag{2.7}$$

and that

$$\int_{K\cap K_j} f\,d\lambda \geqslant p_{i(j)}\mu_{n,i(j)}(K\cap K_j)\inf_{x\in K\cap K_j} f(x) = \mu(K\cap K_j)\inf_{x\in K\cap K_j} f(x). \tag{2.8}$$

Finally combining (2.6), (2.7) and (2.8) show that

$$\begin{aligned}\left|\int_{K\cap K_j} f\,d\mu - \int_{K\cap K_j} f\,d\lambda\right| &\leqslant \mu(K\cap K_j)\Bigl(\sup_{x\in K\cap K_j} f(x) - \inf_{x\in K\cap K_j} f(x)\Bigr)\\ &\leqslant \mu(K\cap K_j)\operatorname{diam}(K\cap K_j).\end{aligned} \tag{2.9}$$

It now follows from (2.5) and (2.9) that

$$\begin{aligned}L(\mu,\lambda) &\leqslant \sup_{f\in \mathrm{Lip}(K)} \sum_j \mu(K\cap K_j)\operatorname{diam}(K\cap K_j)\\ &\leqslant \varepsilon\sum_j \mu(K\cap K_j) = \varepsilon\mu\Bigl(K\cap\bigcup_j K_j\Bigr) = \varepsilon.\end{aligned}$$

This completes the proof. □

2.3. The set Δ^{u} is residual

In this section we prove that the set Δ^{u} is residual. For a real number t write

$$\Delta_t^{\mathrm{u}} = \bigl\{\mu\in\mathcal{P}(K) \mid \overline{D}_\mu(q)\leqslant t(1-q) \text{ for all } q\in[0,1]\bigr\}.$$

Since

$$\Delta^{\mathrm{u}} = \bigcap_{\substack{t\in\mathbb{Q}\\ \bar{s}<t}} \Delta_t^{\mathrm{u}},$$

it clearly suffices to prove that the set Δ_t^{u} is residual for each rational number t with $\bar{s}<t$. Therefore fix a rational number t with $\bar{s}<t$. To prove that the set Δ_t^{u} is residual it clearly suffices to construct a set $M^{\mathrm{u}}\subseteq\mathcal{P}(K)$ satisfying the following three conditions:

(1) $M^{\mathrm{u}}\subseteq\Delta_t^{\mathrm{u}}$;
(2) M^{u} is dense in $\mathcal{P}(K)$;
(3) M^{u} is $\mathcal{G}_\delta$.

Put

$$\begin{aligned}\Lambda^{\mathrm{u}} = \bigl\{\lambda\in\mathcal{P}(K) \mid {}&\text{there exists } x_0\in K \text{ and } r_0>0 \text{ such that}\\ &\overline{\dim}_{\mathrm{B}}\bigl(K\cap B(x_0,r_0)\bigr)<t \text{ and } \lambda\bigl(B(x_0,r_0/2)\bigr)>0\bigr\}.\end{aligned}$$

Hence, for $\lambda \in \Lambda^{\mathrm{u}}$ there exists $x_0 \in K$ and $r_0 > 0$ such that $\overline{\dim}_{\mathrm{B}}(K \cap B(x_0, r_0)) < t$ and $\lambda(B(x_0, \frac{r_0}{2})) > 0$; we now write $r_\lambda = \frac{r_0}{4}\lambda(B(x_0, \frac{r_0}{2}))$. Put

$$M^{\mathrm{u}} = \bigcup_{\lambda \in \Lambda^{\mathrm{u}}} B(\lambda, r_\lambda).$$

Below we show that the set M^{u} has the following three properties: (1) $M^{\mathrm{u}} \subseteq \Delta_t^{\mathrm{u}}$, (2) M^{u} is dense in $\mathcal{P}(K)$, and (3) M^{u} is $\mathcal{G}_\delta$. The set M^{u} is clearly $\mathcal{G}_\delta$, and it thus suffices to show that $M^{\mathrm{u}} \subseteq \Delta_t^{\mathrm{u}}$ and that M^{u} is dense in $\mathcal{P}(K)$. This is done in Propositions 2.3.2 and 2.3.3.

Lemma 2.3.1. *Let $\mu \in \mathcal{P}(K)$ and $E \subseteq K$ with $\mu(E) > 0$. Then*

$$\overline{D}_\mu(q) \leqslant (1-q)\overline{\dim}_{\mathrm{B}}(E)$$

for all $q \in [0, 1]$.

Proof. For a positive real number $r > 0$, let $N_r(E)$ denote the smallest number of balls of radius equal to r that is needed to cover the set E. Then $\overline{\dim}_{\mathrm{B}}(E) = \limsup_{r \searrow 0} \frac{\log N_r(E)}{-\log r}$, cf. [2]. We will now show that

$$I_\mu(2r; q) \leqslant \mu(E)^q \frac{1}{N_r(E)^{q-1}} \tag{2.10}$$

for all $r > 0$. Therefore fix $r > 0$. For brevity write $N = N_r(E)$. We can thus choose balls $B(x_1, r), \ldots, B(x_N, r)$ such that $E \subseteq \bigcup_i B(x_i, r)$. Put $E_1 = B(x_1, r)$ and $E_i = B(x_i, r) \setminus \bigcup_{j=1}^{i-1} B(x_j, r)$ for $i = 2, \ldots, N$. Next observe that if $x \in E_i$, then $E_i \subseteq B(x, 2r)$. We conclude from this that

$$\begin{aligned} I_\mu(2r; q) &= \int \mu\big(B(x, 2r)\big)^{q-1}\, d\mu = \sum_i \int_{E_i} \mu\big(B(x, 2r)\big)^{q-1}\, d\mu \\ &\leqslant \sum_i \int_{E_i} \mu(E_i)^{q-1}\, d\mu = \sum_i \mu(E_i)^q. \end{aligned} \tag{2.11}$$

As $q \in [0, 1]$, the function $t \to t^q$ is concave, and Jensen's inequality therefore implies that

$$\begin{aligned} \sum_i \mu(E_i)^q &= N \sum_i \frac{1}{N}\mu(E_i)^q \leqslant N\bigg(\sum_i \frac{1}{N}\mu(E_i)\bigg)^q = \frac{1}{N^{q-1}}\bigg(\sum_i \mu(E_i)\bigg)^q \\ &= \frac{1}{N^{q-1}}\mu\bigg(\bigcup_i E_i\bigg)^q = \frac{1}{N^{q-1}}\mu(E)^q. \end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12) yields (2.10). This completes the proof of (2.10).

Since $\mu(E) > 0$, the desired conclusion now follows from (2.10) by taking logarithms and dividing by $-\log r$. □

Proposition 2.3.2. *We have $M^{\mathrm{u}} \subseteq \Delta_t^{\mathrm{u}}$.*

Proof. Let $\mu \in M^{\mathrm{u}}$. We can thus choose $\lambda \in \Lambda^{\mathrm{u}}$ such that $L(\mu, \lambda) \leqslant r_\lambda$ where $r_\lambda = \frac{r_0}{4}\lambda(B(x_0, \frac{r_0}{2}))$ for some $x_0 \in K$ and $r_0 > 0$ with $\overline{\dim}_{\mathrm{B}}(K \cap B(x_0, r_0)) < t$ and $\lambda(B(x_0, \frac{r_0}{2})) > 0$. It now follows that (using (2.3))

$$\begin{aligned}\mu\bigl(K \cap B(x_0, r_0)\bigr) &= \int 1_{B(x_0,r_0)}\, d\mu \geqslant \int \frac{f_{x_0,\frac{r_0}{2}}}{\frac{r_0}{2}}\, d\mu \geqslant \frac{2}{r_0}\left(-L(\lambda,\mu) + \int f_{x_0,\frac{r_0}{2}}\, d\lambda\right)\\ &\geqslant \frac{2}{r_0}\left(-r_\lambda + \int_{B(x_0,\frac{r_0}{2})} f_{x_0,\frac{r_0}{2}}\, d\lambda\right) \geqslant \frac{2}{r_0}\left(-r_\lambda + \frac{r_0}{2}\lambda\left(B\left(x_0, \frac{r_0}{2}\right)\right)\right)\\ &= \frac{1}{2}\lambda\left(B\left(x_0, \frac{r_0}{2}\right)\right).\end{aligned}$$

This shows that $\mu(K \cap B(x_0, r_0)) > 0$, and we therefore infer from Lemma 2.3.1 that $t(1-q) \geqslant \overline{\dim}_{\mathrm{B}}(K \cap B(x_0, r_0))(1-q) \geqslant \overline{D}_\mu(q)$ for all $q \in [0, 1]$. □

Proposition 2.3.3. *The set M^{u} is dense in $\mathcal{P}(K)$.*

Proof. Let $\mu \in \mathcal{P}(K)$ and $0 < \varepsilon < 1$. Since $\bar{s} < t$, there exist $x_0 \in K$ and $r_0 > 0$ such that $\overline{\dim}_{\mathrm{B}}(K \cap B(x_0, r_0)) < t$. Now put $\lambda = \frac{\varepsilon}{2}\delta_{x_0} + (1 - \frac{\varepsilon}{2})\mu$. Since $\lambda(B(x_0, \frac{r_0}{2})) \geqslant \frac{\varepsilon}{2} > 0$, we conclude that $\lambda \in \Lambda^{\mathrm{u}} \subseteq M^{\mathrm{u}}$. Also,

$$L(\mu,\lambda) = \sup_{f \in \mathrm{Lip}(K)} \left| \int f\, d\mu - \int f\, d\lambda \right| = \sup_{f \in \mathrm{Lip}(K)} \frac{\varepsilon}{2} \left| \int f\, d\mu - f(x_0) \right| \leqslant \sup_{f \in \mathrm{Lip}(K)} \varepsilon = \varepsilon.$$

This shows that M^{u} is dense in $\mathcal{P}(K)$. □

Proof of Theorem 1. (1) It follows immediately from Lemma 2.3.1 that $\overline{D}_\mu(q) \leqslant s(1-q)$ for all measures $\mu \in \mathcal{P}(K)$ and all $q \in [0, 1]$. Also, $\mu(B(x,r))^{q-1} \geqslant 1$ for all measures $\mu \in \mathcal{P}(K)$ and all $q \in [0, 1]$, whence $I_\mu(r,q) = \int \mu(B(x,r))^{q-1}\, d\mu(x) \geqslant 1$. This clearly implies that $\overline{D}_\mu(q) \geqslant 0$ for all $\mu \in \mathcal{P}(K)$ and all $q \in [0, 1]$.

(2) Recall the definitions of the sets Δ^{u} and Δ^{u} in (2.1). The results in Sections 2.2 and 2.3 show that the sets Δ^{u} and Δ^{u} are residual. We therefore conclude that the set $\Delta^{\mathrm{u}} \cap \Delta^{\mathrm{u}}$ is residual. □

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